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Weighted Sobolev spaces: Markov-type inequalities and duality

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Abstract Weighted Sobolev spaces play a main role in the study of Sobolev orthogonal polynomials. The aim of this paper is to prove several important properties of weighted Sobolev spaces: separability, reflexivity, uniform convexity, duality and Markov-type inequalities.

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1 Introduction

Let us consider $1 \le p < \infty$ and $k+1$ measures $\mu = (\mu_0, \dots, \mu_k)$ in $\mathbb{R}$. We define the *Sobolev norm*

$$\|f\|_{W^{k,p}(\mu)} := \left(\sum_{j=0}^{k} \left\| f^{(j)} \right\|^p_{L^p(\mu_j)} \right)^{1/p}$$

for appropriate functions f (e.g., polynomials if $\mu_0, \dots, \mu_k$ have compact support). The Sobolev space associated to this norm (see the precise definition in Sect. 3) has been very useful in Approximation Theory, but the basic properties of these Sobolev spaces have not been studied so far. The aim of this paper is to prove several important properties of weighted Sobolev spaces: separability, reflexivity, uniform convexity, duality and Markov-type inequalities.

Given a norm on the linear space $\mathbb{P}$ of polynomials with real coefficients, the so-called Markov-type inequalities are estimates connecting the norm of derivatives of a polynomial with the norm of the polynomial itself. These inequalities are interesting by themselves and play a fundamental role in the proof of many inverse theorems in polynomial approximation theory (cf. [25,29] and the references therein).

It is well known that for every polynomial P of degree at most n, the Markov inequality

$$\left\| P' \right\|_{L^\infty([-1,1])} \le n^2 \, \| P \|_{L^\infty([-1,1])}$$

holds and it is optimal since you have equality for the Chebyshev polynomials of the first kind.

The above inequality has been extended to the p norm ($p \ge 1$) in [15]. For every polynomial P of degree at most n their result reads

$$\left\| P' \right\|_{L^p([-1,1])} \le C(n,p) n^2 \, \| P \|_{L^p([-1,1])}$$

where the value of $C(n,p)$ is explicitly given in terms of p and n. Indeed, you have a bound $C(n,p) \le 6e^{1+1/e}$ for $n > 0$ and $p \ge 1$. In [12] admissible values for $C(n,p)$ and some computational results for $p = 2$ are deduced. Notice that for any $p > 1$ and every polynomial P of degree at most n

$$\left\| P' \right\|_{L^p([-1,1])} \le C n^2 \| P \|_{L^p([-1,1])},$$

where C is explicitly given and it is less than the constant $C(n,p)$ in [15].

On the other hand, using matrix analysis, in [8] it is proved that the exact value of $C(n,2)$ is the greatest singular value of the matrix $A_n = [a_{j,k}]_{0 \le j \le n-1, 0 \le k \le n}$, where

$a_{j,k} = \int_{-1}^{1} p_j'(x) p_k(x) dx$ and $\{p_n\}_{n=0}^{\infty}$ is the sequence of orthonormal Legendre polynomials. A simple proof of this result, with an interpretation of the sharp constant $C(n, 2)$ as the largest positive zero of a polynomial as well as an explicit expression of the extremal polynomial (the polynomial such that the inequality becomes an equality) in the L^2-Markov inequality appears in [16].

If you consider weighted L^2-spaces, the problem becomes more difficult. For instance, let $\|\cdot\|_{L^2((a,b),w)}$ be a weighted L^2-norm on $\mathbb{P}$, given by

$$\|P\|_{L^2((a,b),w)} = \left(\int_a^b |P(x)|^2 w(x) dx\right)^{1/2},$$

where w is an integrable function on (a, b), $-\infty \le a < b \le \infty$, such that $w > 0$ a.e. on (a, b) and all moments

$$r_n := \int_a^b x^n w(x) dx, \quad n \ge 0,$$

are finite. It is clear that there exists a constant $\gamma_n = \gamma_n(a, b, w)$ such that

$$\left\|P'\right\|_{L^2((a,b),w)} \le \gamma_n \|P\|_{L^2((a,b),w)}, \quad \text{for all } P \in \mathbb{P}_n, \tag{1.1}$$

where $\mathbb{P}_n$ is the space of polynomials with real coefficients of degree at most n. Indeed, the sharp constant is the greatest singular value of the matrix $B_n = [b_{j,k}]_{0\le j\le n-1, 0\le k\le n}$, where $b_{j,k} = \int_{-1}^{1} p_j'(x) p_k(x) w(x) dx$ and $\{p_n\}_{n=0}^{\infty}$ is the orthonormal polynomial system with respect to the positive measure $w(x)dx$. Thus, from a computational point of view you need to find the connection coefficients between the sequences $\{p_n'\}_{n=0}^{\infty}$ and $\{p_n\}_{n=0}^{\infty}$ in order to proceed with the computation of the matrix, and in a second step, to find the greatest singular value of the matrix B_n. Notice that for classical weights (Jacobi, Laguerre and Hermite), such connection coefficients can be found in a simple way.

Mirsky [27] showed that the best constant $\gamma_n^* := \sup_{P\in\mathbb{P}_n}\{||P'||_{L^2((a,b),w)} : ||P||_{L^2((a,b),w)} = 1\}$ in (1.1) satisfies

$$\gamma_n^* \le \left(\sum_{\nu=1}^{n} \nu \left\|p_\nu'\right\|_{L^2((a,b),w)}^2\right)^{1/2}. \tag{1.2}$$

Notice that the main interest of the above result is however qualitative, since the bound specified by (1.2) can be very crude. In fact, when $w(x) = e^{-x^2}$ on $(-\infty, \infty)$, the estimate (1.2) becomes

$$\gamma_n^* \le \left(\sum_{\nu=1}^{n} 2\nu^2\right)^{1/2} = \sqrt{\frac{1}{3}n(n+1)(2n+1)} = O\big(n^{3/2}\big).$$

The contrast between this estimate and the classic result of Schmidt [40], which establishes $\gamma_n^* = \sqrt{2n}$, is evident.

Also, when we consider the weighted L^2-norm associated with the Laguerre weight $w(x) := x^\alpha e^{-x}$ in $[0, \infty)$, with $a - 1$, the results in [3] give the following inequality

$$\left\| P' \right\|_{L^2(w)} \leq C_\alpha n \left\| P \right\|_{L^2(w)}, \quad \text{for all } P \in \mathbb{P}_n. \tag{1.3}$$

Notice that the nature of the extremal problems associated to the inequalities (1.1) and (1.3) is different, since in the first case the constant on the right-hand side of (1.1) depends on n, while in the second one the multiplicative constant C_α on the right-hand side of (1.3) is independent of n.

There exist a lot of results on Markov-type inequalities (see, e.g. [10,11,25], and the references therein). In connection with the research in the field of the weighted approximation by polynomials, Markov-type inequalities have been proved for various weights, norms, sets over which the norm is taken (see, e.g. [24] and the references therein) and more recently, the study of asymptotic behavior of the sharp constant involved in some kind of these inequalities have been done in [3] for Hermite, Laguerre and Gegenbauer weights and in [4] for Jacobi weights with parameters satisfying some constraints.

On the other hand, a similar problem connected with the Markov–Bernstein inequality has been analyzed in [13] when you try to determine the sharp constant $C(n, m; w)$ such that

$$\left\| A^{m/2} P^{(m)} \right\|_{L^2((a,b),w)} \leq C(n, m; w) \| P \|_{L^2((a,b),w)}, \quad \text{for all } P \in \mathbb{P}_n. \tag{1.4}$$

Here w is a classical weight satisfying a Pearson equation $(A(x)w(x))' = B(x)w(x)$ and A, B are polynomials of degree at most 2 and 1, respectively.

An analogue of the Markov–Bernstein inequality for linear operators T from $\mathbb{P}_n$ into $\mathbb{P}$ has been studied in [18] in terms of singular values of matrices. Some illustrative examples when T is either the derivative or the difference operator and you deal with some classical weights (Laguerre, Gegenbauer in the first case, Charlier, Meixner in the second one) are shown. Another recent application of Markov–Bernstein-type inequalities can be found in [5].

With these ideas in mind, one of the authors of the present paper posed in 2008 during a conference on Constructive Theory of Functions held in Campos do Jordão, Brazil, the following problem: Find the analogous of Markov-type inequalities in the setting of weighted Sobolev spaces. A partial answer of this problem was given in [29], considering an extremal problem with similar conditions to those given by Mirsky, and following the scheme of Kwon and Lee [18], mainly.

Our main aim is the study of several properties involving inequalities in weighted Sobolev spaces.

The first part of this paper is devoted to provide some Markov–Bernstein-type inequalities based on the adequate use of inequalities of kind (1.3) [3,10,40], in the setting of weighted Sobolev spaces, when the considered weights are generalized classical weights.

In the second part we study other basic facts about Sobolev spaces with respect to measures: separability, reflexivity, uniform convexity and duality, which to the best of our knowledge are not available in the current literature. In order to obtain these properties we work with other inequalities on weighted Sobolev spaces (e.g., Muckenhoupt inequality).

These Sobolev spaces appear in a natural way and are a very useful tool when we study the asymptotic behavior of Sobolev orthogonal polynomials (see [7,19–21,33–35,37,39]). In particular, the completeness of these spaces has been shown very useful; it is a remarkable fact that it took 6 years to prove this natural property (see [2,36,37,39]).

The outline of the paper is as follows. The first part of Sect. 2 provides some short background about Markov-type inequalities in L^2 spaces with classical weights and the second one deals with a Markov-type inequality corresponding to each weighted Sobolev norm with respect to these classical weights and to some generalized weights (see Theorem 2.1). Section 3 contains definitions and a discussion about the appropriate vector measures which we will need in order to get completeness of our Sobolev spaces with respect to measures. Finally, Sect. 4 contains some basic results on Sobolev spaces with respect to the vector measures defined in the previous section (see Theorems 4.2 and 4.3): separability, reflexivity, uniform convexity and duality.

2 Markov-type inequalities in Sobolev spaces with weights

The following proposition summarizes the Markov-type inequalities in L^2 spaces with classical weights, which will be used in the sequel. Recall that we denote by $\mathbb{P}_n$ the linear space of polynomials with real coefficients and degree less than or equal to n.

Proposition 2.1 *The following inequalities are satisfied.*

(1) *Laguerre case* [3] *(see also* [9]*):*

$$\left\| P' \right\|_{L^2(w)} \leq C_\alpha n \left\| P \right\|_{L^2(w)},$$

where $w(x) := x^\alpha e^{-x}$ *in* $[0, \infty)$*,* $\alpha > -1$ *and* $P \in \mathbb{P}_n$*.*

(2) *Generalized Hermite case* [10,40]*:*

$$\left\| P' \right\|_{L^2(w)} \leq \sqrt{2n} \left\| P \right\|_{L^2(w)},$$

where $w(x) := |x|^\alpha e^{-x^2}$ *in* $\mathbb{R}$*,* $\alpha \geq 0$ *and* $P \in \mathbb{P}_n$*.*

(3) *Jacobi case* [11]*:*

$$\left\| P' \right\|_{L^2(w)} \leq C_{\alpha,\beta}\, n^2 \left\| P \right\|_{L^2(w)},$$

where $w(x) := (1-x)^\alpha (1+x)^\beta$ *in* $[-1, 1]$*,* $\alpha, \beta > -1$ *and* $P \in \mathbb{P}_n$*.*

The multiplicative constants C_α *and* $C_{\alpha,\beta}$ *are independent of* n*.*

Remark 2.1 The results in [3] and [11] are different from (1) and (3) in Proposition 2.1. However, one can deduce Proposition 2.1 from these results.

For instance, by [3] we know that if

$$M_{\alpha,n} := \sup_{P\in\mathbb{P}_n} \frac{\|P'\|_{L^2(w)}}{\|P\|_{L^2(w)}}$$

for $w(x) := x^\alpha e^{-x}$ in $[0,\infty)$ and $\alpha > -1$, then there exists a positive constant k_α such that

$$M_{\alpha,n} = \frac{n}{k_\alpha}\big(1 + O(1)\big).$$

Hence, for each fixed $\alpha > -1$,

$$M_\alpha := \sup_n \frac{M_{\alpha,n}}{n} < \infty \quad \text{and} \quad \|P'\|_{L^2(w)} \le M_\alpha\, n\, \|P\|_{L^2(w)}$$

for every $P \in \mathbb{P}_n$. Also, this inequality can be deduced from [9].

By [11] we know that if

$$M_{\alpha,\beta,n} := \sup_{P\in\mathbb{P}_n} \frac{\|P'\|_{L^2(w)}}{\|P\|_{L^2(w)}}$$

for $w(x) := (1-x)^\alpha(1+x)^\beta$ in $[-1,1]$ and $\alpha,\beta > -1$, then

$$M_{\alpha,\beta,n} \le N_{\alpha,\beta,n} := \sqrt{\frac{n(n+1)(\alpha+\beta+2)(\alpha+\beta+n+1)(\alpha+\beta+n+2)}{8(\alpha+1)(\beta+1)}}\,.$$

Hence, for each fixed $\alpha,\beta > -1$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{N_{\alpha,\beta,n}}{n^2} &= \lim_{n\to\infty} \frac{\sqrt{\frac{n(n+1)(\alpha+\beta+2)(\alpha+\beta+n+1)(\alpha+\beta+n+2)}{8(\alpha+1)(\beta+1)}}}{n^2} \\
&= \sqrt{\frac{(\alpha+\beta+2)}{8(\alpha+1)(\beta+1)}} < \infty, \\
N_{\alpha,\beta} &:= \sup_n \frac{N_{\alpha,\beta,n}}{n^2} < \infty, \quad M_{\alpha,\beta,n} \le N_{\alpha,\beta,n} \le N_{\alpha,\beta}\, n^2, \\
\|P'\|_{L^2(w)} &\le N_{\alpha,\beta}\, n^2\, \|P\|_{L^2(w)}
\end{aligned}$$

for every $P \in \mathbb{P}_n$.

We are interested in obtaining bounds such as a constant (independent of n) multiplied by a power of n.

In Theorem 2.1 below we extend these results to the context of weighted Sobolev spaces. We want to remark that the proof provides explicit expressions for the involved constants.

Theorem 2.1 *The following inequalities are satisfied.*

(1) *Laguerre–Sobolev case:*

$$\|P'\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)} \le C_\alpha n \, \|P\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)},$$

where $w(x) := x^\alpha e^{-x}$ *in* $[0,\infty)$, $\alpha > -1$, $\lambda_1,\ldots,\lambda_k \ge 0$, $P \in \mathbb{P}_n$ *and* C_α *is the same constant as in Proposition* 2.1 (1).

(2) *Generalized Hermite–Sobolev case:*

$$\|P'\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)} \le \sqrt{2n} \, \|P\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)},$$

where $w(x) := |x|^\alpha e^{-x^2}$ *in* $\mathbb{R}$, $\alpha \ge 0$, $\lambda_1,\ldots,\lambda_k \ge 0$ *and* $P \in \mathbb{P}_n$.

(3) *Jacobi–Sobolev case:*

$$\|P'\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)} \le C_{\alpha,\beta} \, n^2 \, \|P\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)},$$

where $w(x) := (1-x)^\alpha(1+x)^\beta$ *in* $[-1,1]$, $\alpha,\beta > -1$, $\lambda_1,\ldots,\lambda_k \ge 0$, $P \in \mathbb{P}_n$ *and* $C_{\alpha,\beta}$ *is the constant in Proposition 2.1* (3).

(4) *Let us consider the generalized Jacobi weight* $w(x) := h(x)\Pi_{j=1}^r |x-c_j|^{\gamma_j}$ *in* $[a,b]$ *with* $c_1,\ldots,c_r \in \mathbb{R}$, $\gamma_1,\cdots,\gamma_r \in \mathbb{R}$, $\gamma_j > -1$ *when* $c_j \in [a,b]$, *and* h *a measurable function satisfying* $0 < m \le h \le M$ *in* $[a,b]$ *for some constants* m, M. *Then we have*

$$\begin{aligned}\|P'\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)} \le\ & C_1(a,b,c_1,\ldots,c_r,\gamma_1,\ldots,\gamma_r,m,M) \\ & \times n^2 \, \|P\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)},\end{aligned}$$

for every $\lambda_1,\ldots,\lambda_k \ge 0$ *and* $P \in \mathbb{P}_n$.

(5) *Consider now the generalized Laguerre weight* $w(x) := h(x)\Pi_{j=1}^r |x-c_j|^{\gamma_j} e^{-x}$ *in* $[0,\infty)$ *with* $c_1 < \cdots < c_r$, $c_r \ge 0$, $\gamma_1,\ldots,\gamma_r \in \mathbb{R}$, $\gamma_j > -1$ *when* $c_j \ge 0$, *and* h *a measurable function satisfying* $0 < m \le h \le M$ *in* $[0,\infty)$ *for some constants* m, M.

(5.1) *If* $\sum_{j=1}^{r-1} \gamma_j = 0$, *then*

$$\begin{aligned}\|P'\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)} \le\ & C_2(c_1,\ldots,c_r,\gamma_1,\ldots,\gamma_r,m,M) \\ & \times n^2 \, \|P\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)},\end{aligned}$$

for every $\lambda_1,\ldots,\lambda_k \ge 0$ *and* $P \in \mathbb{P}_n$.

(5.2) *Assume that* $c_1 < \cdots < c_r$ *and* $\sum_{j=1}^r \gamma_j > -1$. *Let* $r_0 := \min\{1 \le j \le r \mid c_j \ge 0\}$, $\gamma'_{r_0-1} := \gamma'_{r+1} := 0$ *and* $\gamma'_j := \gamma_j$ *for every* $r_0 \le j \le r$. *Assume that* $\max\{\gamma'_j, \gamma'_{j+1}\} \ge -1/2$ *for every* $r_0 - 1 \le j \le r$. *Then we have*

$$\begin{aligned}\|P'\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)} \le\ & C'_2(c_1,\ldots,c_r,\gamma_1,\ldots,\gamma_r,m,M) \\ & \times n^{a'} \, \|P\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)},\end{aligned}$$

for every $\lambda_1, \dots, \lambda_k \geq 0$ *and* $P \in \mathbb{P}_n$, *where*

$$a' := \max\left\{2\,,\ \frac{b'+2}{2}\right\}, \quad b' := \max_{r_0-1\leq j\leq r}\left(\gamma_j' + \gamma_{j+1}' + |\gamma_j' - \gamma_{j+1}'| + 2\right).$$

(6) *Let us consider the generalized Hermite weight* $w(x) := h(x)\Pi_{j=1}^r |x-c_j|^{\gamma_j} e^{-x^2}$ *in* $\mathbb{R}$ *with* $c_1 < \cdots < c_r$, $\gamma_1, \dots, \gamma_r > -1$ *with* $\sum_{j=1}^r \gamma_j \geq 0$ *and* h *a measurable function satisfying* $0 < m \leq h \leq M$ *in* $\mathbb{R}$ *for some constants* m, M. *Define* $\gamma_0 := \gamma_{r+1} := 0$ *and assume that* $\max\{\gamma_j, \gamma_{j+1}\} \geq -1/2$ *for every* $0 \leq j \leq r$. *Then we have*

$$\|P'\|_{W^{k,2}(w,\lambda_1 w,\dots,\lambda_k w)} \leq C_3(c_1, \dots, c_r, \gamma_1, \dots, \gamma_r, m, M) \times n^a\, \|P\|_{W^{k,2}(w,\lambda_1 w,\dots,\lambda_k w)},$$

for every $\lambda_1, \dots, \lambda_k \geq 0$ *and* $P \in \mathbb{P}_n$, *where*

$$a := \max\left\{2\,,\ \frac{b+1}{2}\right\}, \quad b := \max_{0\leq j\leq r}\left(\gamma_j + \gamma_{j+1} + |\gamma_j - \gamma_{j+1}| + 2\right).$$

In each case the multiplicative constants depend just on the specified parameters (in particular, they do not depend on n*).*

Remark 2.2 Note that (4), (5), and (6) are new results in the classical (non-Sobolev) context (taking $\lambda_1 = \cdots = \lambda_k = 0$). In (5.2), there is no hypothesis on $\sum_{j=1}^{r-1} \gamma_j$.

Remark 2.3 In the inequalities in (1), (2), (3), (4) and (5.1) appear powers of n with exponent at most 2. One could expect to obtain the power n^2 in (5.2) and (6) (we will need to bound in the proof some norms in compact intervals with Jacobi weights, and then in those computations the exponent 2 comes up). This is the case if $b' \leq 2$ and $b \leq 3$, respectively. Otherwise, the argument in the proof gives exponents a' and a, respectively. These exponents appear as a consequence of Lupaş' inequality [22], but we need it (or some similar inequality) in order to obtain, for every $P \in \mathbb{P}_n$,

$$\int_{-1}^{1} |P(x)|^2 dx \leq K_n(\alpha, \beta) \int_{-1}^{1} |P(x)|^2 (1-x)^\alpha (1+x)^\beta dx.$$

If $\alpha \leq 0$ and $\beta \leq 0$, then we obtain this inequality with a constant independent of n (note that we have the continuous inclusion $L^2([-1,1], (1-x)^\alpha(1+x)^\beta dx) \subseteq L^2([-1,1], dx)$ since $\inf_{x\in(-1,1)}(1-x)^\alpha(1+x)^\beta > 0$). However, if $\alpha > 0$ or $\beta > 0$, then it is clear that $K_n(\alpha, \beta)$ must grow with n, since the difference $L^2([-1,1], (1-x)^\alpha(1+x)^\beta dx)\setminus L^2([-1,1], dx)$ is non-empty.

Proof First of all, note that if the inequality

$$\|P'\|_{L^2(w)} \leq C(n, w)\, \|P\|_{L^2(w)}$$

holds for every polynomial $P \in \mathbb{P}_n$ and some fixed weight w, then we have

$$\|P^{(j+1)}\|_{L^2(\lambda w)} \le C(n, w)\, \|P^{(j)}\|_{L^2(\lambda w)} \tag{2.1}$$

for every polynomial $P \in \mathbb{P}_n$ and every $\lambda \ge 0$. Consequently, for the weighted Sobolev norm on $\mathbb{P}$

$$\|P\|_{W^{k,2}(w,\lambda_1 w,\dots,\lambda_k w)} := \left(\|P\|^2_{L^2(w)} + \sum_{j=1}^{k} \|P^{(j)}\|^2_{L^2(\lambda_j w)} \right)^{1/2}, \quad \lambda_1, \dots, \lambda_k \ge 0,$$

we have

$$\|P'\|_{W^{k,2}(w,\lambda_1 w,\dots,\lambda_k w)} \le C(n, w)\, \|P\|_{W^{k,2}(w,\lambda_1 w,\dots,\lambda_k w)}, \tag{2.2}$$

for every polynomial $P \in \mathbb{P}_n$ and every $\lambda_1, \dots, \lambda_k \ge 0$.

Thus, (1), (2) and (3) hold.

In order to prove (4), note that, for each $a_1 < a_2$, by using Proposition 2.1 (3) and the affine transformation

$$Tx = \frac{2x - a_1 - a_2}{a_2 - a_1}$$

(which maps $[a_1, a_2]$ onto $[-1, 1]$), we obtain

$$\|P'\|_{L^2(w)} \le C(a_1, a_2, \alpha, \beta)\, n^2\, \|P\|_{L^2(w)},$$

for the weight $w(x) := (a_2 - x)^{\alpha}(x - a_1)^{\beta}$ in $[a_1, a_2]$ and every polynomial $P \in \mathbb{P}_n$.

Without loss of generality we can assume that $a \le c_1 < \cdots < c_r \le b$, since otherwise we can consider

$$w(x) = \tilde{h}(x) \prod_{\substack{1\le j\le r\\ c_j\in[a,b]}} |x - c_j|^{\gamma_j}, \quad \tilde{h}(x) := h(x) \prod_{\substack{1\le j\le r\\ c_j\notin[a,b]}} |x - c_j|^{\gamma_j}.$$

If we define $c_0 := a$, $c_{r+1} := b$ and $\gamma_0 := \gamma_{r+1} := 0$, then we can write $w(x) = h(x)\Pi_{j=0}^{r+1}|x - c_j|^{\gamma_j}$. Denote by h_j the function

$$h_j(x) := \frac{w(x)}{|x - c_j|^{\gamma_j}\,|x - c_{j+1}|^{\gamma_{j+1}}},$$

for $0 \le j \le r$. It is clear that there exist positive constants m_j, M_j (depending just on $m, M, c_1, \dots, c_{j-1}, c_{j+2}, \dots, c_r, \gamma_1, \dots, \gamma_{j-1}, \gamma_{j+2}, \dots, \gamma_r$), with $m_j \le h_j(x) \le M_j$ for every $x \in [c_j, c_{j+1}]$.

Hence, for $P \in \mathbb{P}_n$, we have

$$
\begin{aligned}
\|P'\|_{L^2([c_j,c_{j+1}],w)} &= \left(\int_{c_j}^{c_{j+1}} \left|P'(x)\right|^2 \, h_j(x) |x-c_j|^{\gamma_j} |x-c_{j+1}|^{\gamma_{j+1}} \, dx \right)^{1/2} \\
&\le \sqrt{M_j} \left(\int_{c_j}^{c_{j+1}} \left|P'(x)\right|^2 \, |x-c_j|^{\gamma_j} |x-c_{j+1}|^{\gamma_{j+1}} \, dx \right)^{1/2} \\
&\le \sqrt{M_j} \, C\left(c_j, c_{j+1}, \gamma_j, \gamma_{j+1}\right) \, n^2 \\
&\quad \times \left(\int_{c_j}^{c_{j+1}} |P(x)|^2 \, |x-c_j|^{\gamma_j} |x-c_{j+1}|^{\gamma_{j+1}} \, dx \right)^{1/2} \\
&\le \sqrt{M_j} \, C(c_j, c_{j+1}, \gamma_j, \gamma_{j+1}) \, n^2 \\
&\quad \times \left(\int_{c_j}^{c_{j+1}} |P(x)|^2 \, |x-c_j|^{\gamma_j} |x-c_{j+1}|^{\gamma_{j+1}} \frac{h_j(x)}{m_j} \, dx \right)^{1/2} \\
&= \sqrt{\frac{M_j}{m_j}} \, C(c_j, c_{j+1}, \gamma_j, \gamma_{j+1}) \, n^2 \, \|P\|_{L^2([c_j,c_{j+1}],w)}.
\end{aligned}
$$

Next, "pasting" several times this last inequality in each subinterval $[c_j, c_{j+1}] \subseteq [a, b]$, $0 \le j \le r$, we obtain

$$
\|P'\|_{L^2(w)} \le C_1(a, b, c_1, \ldots, c_r, \gamma_1, \ldots, \gamma_r, m, M) \, n^2 \, \|P\|_{L^2(w)}
$$

for every polynomial $P \in \mathbb{P}_n$, with

$$
C_1(a, b, c_1, \ldots, c_r, \gamma_1, \ldots, \gamma_r, m, M) := \max_{0 \le j \le r} \sqrt{\frac{M_j}{m_j}} \, C(c_j, c_{j+1}, \gamma_j, \gamma_{j+1}).
$$

Hence, we obtain the case (4) by applying (2.2).

Similarly, for the case (5.1) we can write

$$
w(x) = H_1(x) \prod_{j=1}^{r} |x-c_j|^{\gamma_j},
$$

where $H_1(x) := h(x)e^{-x}$ satisfies $0 < m \, e^{-c_r} \le H_1 \le M$ in $[0, c_r]$. Then the case (4) provides a constant $\hat{C}_1$, which just depends on the appropriate parameters, with

$$
\|P'\|^2_{W^{k,2}([0,c_r],w,\lambda_1 w,\ldots,\lambda_k w)} \le \hat{C}_1^2 \, n^4 \, \|P\|^2_{W^{k,2}([0,c_r],w,\lambda_1 w,\ldots,\lambda_k w)}, \tag{2.3}
$$

for every $\lambda_1, \ldots, \lambda_k \ge 0$ and every polynomial $P \in \mathbb{P}_n$.

Proposition 2.1 (1) gives

$$
\|P'\|_{L^2(w_1)} \le C_\alpha \, n \, \|P\|_{L^2(w_1)},
$$

where $w_1(x) := x^\alpha e^{-x}$ in $[0, \infty)$, $\alpha > -1$ and $P \in \mathbb{P}_n$. Hence, replacing x by $x - c$, we obtain with the same constant C_α

$$\|P'\|_{L^2([c,\infty),(x-c)^\alpha e^{c-x})} \le C_\alpha\, n\, \|P\|_{L^2([c,\infty),(x-c)^\alpha e^{c-x})},$$

for every $c \ge 0$ and $P \in \mathbb{P}_n$. Now, if $w_2(x) := (x-c)^\alpha e^{-x}$, then the previous inequality implies

$$\|P'\|_{L^2([c,\infty),w_2)} \le C_\alpha\, n\, \|P\|_{L^2([c,\infty),w_2)},$$

for every $c \ge 0$ and $P \in \mathbb{P}_n$, and (2.2) gives

$$\|P'\|_{W^{k,2}([c,\infty),w_2,\lambda_1 w_2,\ldots,\lambda_k w_2)} \le C_\alpha\, n\, \|P\|_{W^{k,2}([c,\infty),w_2,\lambda_1 w_2,\ldots,\lambda_k w_2)},$$

for every $c \ge 0$, $\lambda_1, \ldots, \lambda_k \ge 0$ and $P \in \mathbb{P}_n$.

We can write now

$$w(x) = H_2(x)(x - c_r)^{\gamma_r} e^{-x},$$

where $H_2(x) := h(x)\Pi_{j=1}^{r-1}|x-c_j|^{\gamma_j}$ and there exist constants m_2, M_2 with $0 < m_2 \le H_2 \le M_2$ in $[c_r, \infty)$, since $\sum_{j=1}^{r-1} \gamma_j = 0$. Thus,

$$\|P'\|^2_{W^{k,2}([c_r,\infty),w,\lambda_1 w,\ldots,\lambda_k w)} \le C^2_{\gamma_r} \frac{M_2}{m_2}\, n^2\, \|P\|^2_{W^{k,2}([c_r,\infty),w,\lambda_1 w,\ldots,\lambda_k w)}, \tag{2.4}$$

for every $\lambda_1, \ldots, \lambda_k \ge 0$ and $P \in \mathbb{P}_n$.

If we define $\hat{C}_2 := \sqrt{\hat{C}_1^2 + C^2_{\gamma_r} M_2/m_2}$, then (2.3) and (2.4) give

$$\|P'\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)} \le \hat{C}_2\, n^2\, \|P\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)},$$

for every $\lambda_1, \ldots, \lambda_k \ge 0$ and $P \in \mathbb{P}_n$.

Let us prove now (5.2). Define $A := 1 + c_r$. We can write

$$w(x) = H_3(x) \prod_{j=1}^{r} |x - c_j|^{\gamma_j},$$

where $H_3(x) := h(x)e^{-x}$ satisfies $0 < m\, e^{-A} \le H_3 \le M$ in $[0, A]$. Then the case (4) provides a constant C_1, which just depends on the appropriate parameters, with

$$\|P'\|_{W^{k,2}([0,A],w,\lambda_1 w,\ldots,\lambda_k w)} \le C_1\, n^2\, \|P\|_{W^{k,2}([0,A],w,\lambda_1 w,\ldots,\lambda_k w)}, \tag{2.5}$$

for every $\lambda_1, \ldots, \lambda_k \ge 0$ and every polynomial $P \in \mathbb{P}_n$.

Proposition 2.1 (1) gives a constant C_s with

$$\|P'\|^2_{L^2(w_3)} \le C_s^2 n^2\, \|P\|^2_{L^2(w_3)},$$

where $w_3(x) := x^s e^{-x}$, $s := \sum_{j=1}^{r} \gamma_j > -1$ and $P \in \mathbb{P}_n$.

We can write now

$$w(x) = H_4(x)x^s e^{-x} = H_4(x)w_3(x),$$

where $H_4(x) := h(x)x^{-s}\Pi_{j=1}^{r}|x - c_j|^{\gamma_j}$, and there exist constants m_4, M_4 with $0 < m_4 \leq H_4 \leq M_4$ in $[A, \infty)$, since $s = \sum_{j=1}^{r} \gamma_j$. Thus,

$$\begin{aligned} \|P'\|^2_{L^2([A,\infty),w)} &\leq M_4\|P'\|^2_{L^2(w_3)} \leq C_s^2 n^2 M_4 \,\|P\|^2_{L^2(w_3)} \\ &\leq C_s^2 n^2 \frac{M_4}{m_4}\, \|P\|^2_{L^2([A,\infty),\,w)} + C_s^2 n^2 M_4 \,\|P\|^2_{L^2([0,A],\,w_3)}, \end{aligned} \tag{2.6}$$

for every $P \in \mathbb{P}_n$.

Using Lupaş' inequality [22] (see also [26, p. 594]):

$$\begin{aligned} \|P\|_{L^\infty([-1,1])} \leq & \sqrt{\frac{\Gamma(n+\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(q+1)\Gamma(n+q'+1)}\binom{n+q+1}{n}} \\ & \times\sqrt{\int_{-1}^{1} |P(x)|^2(1-x)^\alpha(1+x)^\beta dx}, \end{aligned}$$

for every $P \in \mathbb{P}_n$, where $q = \max(\alpha, \beta) \geq -1/2$ and $q' = \min(\alpha, \beta)$, we obtain that

$$\begin{aligned} 2\|P\|^2_{L^\infty([-1,1])} \leq & \frac{\Gamma(n+\alpha+\beta+2)\Gamma(n+q+2)}{2^{\alpha+\beta}\Gamma(q+1)\Gamma(q+2)\Gamma(n+1)\Gamma(n+q'+1)} \\ & \times \int_{-1}^{1} |P(x)|^2(1-x)^\alpha(1+x)^\beta dx. \end{aligned}$$

Now, taking into account that

$$\lim_{n\to\infty} \frac{\Gamma(n+x)}{\Gamma(n+y)n^{x-y}} = 1, \quad x, y \in \mathbb{R},$$

we get

$$\frac{\Gamma(n+\alpha+\beta+2)\Gamma(n+q+2)}{\Gamma(n+1)\Gamma(n+q'+1)} \sim n^{\alpha+\beta+1}n^{q-q'+1} = n^{\alpha+\beta+|\alpha-\beta|+2}.$$

Consequently, there exists a constant k_1, which just depends on α and β, such that

$$\int_{-1}^{1} |P(x)|^2 dx \leq 2\|P\|^2_{L^\infty([-1,1])} \leq k_1(\alpha, \beta)n^{v(\alpha,\beta)} \int_{-1}^{1} |P(x)|^2(1-x)^\alpha(1+x)^\beta dx,$$

where $v(\alpha, \beta) = \alpha + \beta + |\alpha - \beta| + 2$ and $P \in \mathbb{P}_n$.

Recall that $\max\{\gamma_j', \gamma_{j+1}'\} \geq -1/2$ for every $r_0 - 1 \leq j \leq r$ and

$$b' := \max_{r_0-1 \leq j \leq r} \left(\gamma_j' + \gamma_{j+1}' + |\gamma_j' - \gamma_{j+1}'| + 2\right).$$

Therefore, a similar argument to the one in the proof of (4) gives

$$\begin{aligned}\int_0^A |P(x)|^2\, dx &\leq k_2\, n^{b'} \int_0^A |P(x)|^2 \prod_{j=r_0-1}^{r+1} |x - c_j|^{\gamma_j'}\, dx \\ &= k_2\, n^{b'} \int_0^A |P(x)|^2 \prod_{j=r_0}^{r} |x - c_j|^{\gamma_j}\, dx,\end{aligned}$$

for every polynomial $P \in \mathbb{P}_n$ and some constant k_2 which just depends on $c_{r_0}, \ldots, c_r, \gamma_{r_0}, \ldots, \gamma_r$. Thus,

$$\int_0^A |P(x)|^2\, dx \leq k_3\, n^{b'} \int_0^A |P(x)|^2 \prod_{j=1}^{r} |x - c_j|^{\gamma_j}\, dx,$$

for every polynomial $P \in \mathbb{P}_n$ and some constant k_3 which just depends on $c_1, \ldots, c_r, \gamma_1, \ldots, \gamma_r$.

Hence,

$$\begin{aligned}\|P\|^2_{L^2([0,A],w_3)} &= \int_0^A |P(x)|^2 x^s e^{-x}\, dx \leq A^s \int_0^A |P(x)|^2\, dx \\ &\leq k_3\, n^{b'} A^s \int_0^A |P(x)|^2 \prod_{j=1}^{r} |x - c_j|^{\gamma_j}\, dx \\ &\leq \frac{1}{m} k_3\, n^{b'} A^s e^A \int_0^A |P(x)|^2 h(x) \prod_{j=1}^{r} |x - c_j|^{\gamma_j} e^{-x}\, dx \\ &= \frac{1}{m} k_3\, A^s e^A n^{b'} \|P\|^2_{L^2([0,A],w)},\end{aligned}$$

for every polynomial $P \in \mathbb{P}_n$.

This inequality and (2.6) give

$$\begin{aligned}\|P'\|^2_{L^2([A,\infty),w)} &\leq C_s^2 \frac{M_4}{m_4} n^2 \|P\|^2_{L^2([A,\infty),w)} + C_s^2 \frac{M_4}{m} k_3\, A^s e^A n^{b'+2} \|P\|^2_{L^2([0,A],w)} \\ &\leq k_4\, n^{b'+2} \left(\|P\|^2_{L^2([A,\infty),w)} + \|P\|^2_{L^2([0,A],w)}\right) \\ &= k_4\, n^{b'+2} \|P\|^2_{L^2(w)},\end{aligned}$$

for every polynomial $P \in \mathbb{P}_n$, where

$$k_4 := \max\left\{C_s^2 \frac{M_4}{m_4}, C_s^2 \frac{M_4}{m} k_3 A^s e^A\right\}.$$

Hence,

$$\|P'\|^2_{W^{k,2}([A,\infty),\, w,\lambda_1 w,\dots,\lambda_k w)} \le k_4\, n^{b'+2}\, \|P\|^2_{W^{k,2}(w,\lambda_1 w,\dots,\lambda_k w)},$$

for every $\lambda_1,\dots,\lambda_k \ge 0$ and every polynomial $P \in \mathbb{P}_n$, and (2.5) allows to deduce

$$\|P'\|^2_{W^{k,2}(w,\lambda_1 w,\dots,\lambda_k w)} \le \left(C_1^2 n^4 + k_4\, n^{b'+2}\right)\|P\|^2_{W^{k,2}(w,\lambda_1 w,\dots,\lambda_k w)},$$

for every $\lambda_1,\dots,\lambda_k \ge 0$ and every polynomial $P \in \mathbb{P}_n$. If we define

$$k_5 := \left(C_1^2 + k_4\right)^{1/2},$$

and we recall that

$$a' := \max\left\{2, \frac{b'+2}{2}\right\},$$

then

$$\|P'\|_{W^{k,2}(w,\lambda_1 w,\dots,\lambda_k w)} \le k_5\, n^{a'}\, \|P\|_{W^{k,2}(w,\lambda_1 w,\dots,\lambda_k w)},$$

for every $\lambda_1,\dots,\lambda_k \ge 0$ and every polynomial $P \in \mathbb{P}_n$.

Finally, let us show (6). Define $B := 1 + \max\{|c_1|, |c_r|\}$. We can write

$$w(x) = H_5(x) \prod_{j=1}^{r} |x - c_j|^{\gamma_j},$$

where $H_5(x) := h(x)e^{-x^2}$ satisfies $0 < m\, e^{-B^2} \le H_5 \le M$ in $[-B, B]$. Then the case (4) provides a constant C_1, which just depends on the appropriate parameters, with

$$\|P'\|_{W^{k,2}([-B,B],w,\lambda_1 w,\dots,\lambda_k w)} \le C_1\, n^2\, \|P\|_{W^{k,2}([-B,B],w,\lambda_1 w,\dots,\lambda_k w)}, \tag{2.7}$$

for every $\lambda_1,\dots,\lambda_k \ge 0$ and every polynomial $P \in \mathbb{P}_n$.

Proposition 2.1 (2) gives

$$\|P'\|^2_{L^2(w_4)} \le 2n\, \|P\|^2_{L^2(w_4)},$$

where $w_4(x) := |x|^\alpha e^{-x^2}$, $\alpha := \sum_{j=1}^{r} \gamma_j \ge 0$ and $P \in \mathbb{P}_n$.

We can write now

$$w(x) = H_6(x)|x|^{\alpha} e^{-x^2} = H_6(x) w_4(x),$$

where $H_6(x) := h(x)|x|^{-\alpha} \Pi_{j=1}^r |x - c_j|^{\gamma_j}$, and there exist constants m_6, M_6 with $0 < m_6 \le H_6 \le M_6$ in $(-\infty, -B] \cup [B, \infty)$, since $\alpha = \sum_{j=1}^r \gamma_j$. Thus,

$$\begin{aligned} \|P'\|^2_{L^2((-\infty,-B]\cup[B,\infty),w)} &\le M_6 \|P'\|^2_{L^2(w_4)} \le 2n M_6 \,\|P\|^2_{L^2(w_4)} \\ &\le 2n \,\frac{M_6}{m_6}\, \|P\|^2_{L^2((-\infty,-B]\cup[B,\infty),\, w)} \\ &\quad + 2n M_6 \,\|P\|^2_{L^2([-B,B],\, w_4)}, \end{aligned} \tag{2.8}$$

for every $P \in \mathbb{P}_n$.

Since $\max\{\gamma_j, \gamma_{j+1}\} \ge -1/2$ for every $0 \le j \le r$ and $b := \max_{0 \le j \le r} \big(\gamma_j + \gamma_{j+1} + |\gamma_j - \gamma_{j+1}| + 2\big)$, the argument in the proof of (5.2), using Lupaş' inequality, gives

$$\int_{-B}^{B} |P(x)|^2 \, dx \le k_6 \, n^b \int_{-B}^{B} |P(x)|^2 \prod_{j=1}^{r} |x - c_j|^{\gamma_j} \, dx,$$

for every polynomial $P \in \mathbb{P}_n$ and some constant k_6 which just depends on $c_1, \dots, c_r, \gamma_1, \dots, \gamma_r$.

Hence,

$$\begin{aligned} \|P\|^2_{L^2([-B,B],w_4)} &= \int_{-B}^{B} |P(x)|^2 |x|^{\alpha} e^{-x^2} \, dx \le B^{\alpha} \int_{-B}^{B} |P(x)|^2 \, dx \\ &\le k_6 \, n^b B^{\alpha} \int_{-B}^{B} |P(x)|^2 \prod_{j=1}^{r} |x - c_j|^{\gamma_j} \, dx \\ &\le \frac{1}{m} \, k_6 \, n^b B^{\alpha} e^{B^2} \int_{-B}^{B} |P(x)|^2 h(x) \prod_{j=1}^{r} |x - c_j|^{\gamma_j} e^{-x^2} \, dx \\ &\le \frac{1}{m} \, k_6 \, n^b B^{\alpha} e^{B^2} \|P\|^2_{L^2(w)}, \end{aligned}$$

for every polynomial $P \in \mathbb{P}_n$.

This inequality and (2.8) give

$$\begin{aligned} \|P'\|^2_{L^2((-\infty,-B]\cup[B,\infty),w)} &\le \max\left\{2\,\frac{M_6}{m_6}\, n,\; 2\,\frac{M_6}{m}\, k_6 \, B^{\alpha} e^{B^2} n^{b+1}\right\} \|P\|^2_{L^2(w)} \\ &\le k_7 \, n^{b+1} \, \|P\|^2_{L^2(w)}, \end{aligned}$$

for every polynomial $P \in \mathbb{P}_n$, where

$$k_7 := \max\left\{2\,\frac{M_6}{m_6}\,,\,2\,\frac{M_6}{m}\,k_6\,B^\alpha e^{B^2}\right\}.$$

Hence,

$$\left\|P'\right\|^2_{W^{k,2}((-\infty,-B]\cup[B,\infty),\,w,\lambda_1 w,\ldots,\lambda_k w)} \le k_7\, n^{b+1}\ \|P\|^2_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)},$$

for every $\lambda_1,\ldots,\lambda_k \ge 0$ and every polynomial $P \in \mathbb{P}_n$, and (2.7) allows to deduce

$$\|P'\|^2_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)} \le \left(C_1^2 n^4 + k_7\, n^{b+1}\right)\|P\|^2_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)},$$

for every $\lambda_1,\ldots,\lambda_k \ge 0$ and every polynomial $P \in \mathbb{P}_n$. If we define

$$k_8 := \left(C_1^2 + k_7\right)^{1/2},$$

and we recall that

$$a := \max\left\{2\,,\ \frac{b+1}{2}\right\},$$

then

$$\|P'\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)} \le k_8\, n^a\, \|P\|_{W^{k,2}(w,\lambda_1 w,\ldots,\lambda_k w)},$$

for every $\lambda_1,\ldots,\lambda_k \ge 0$ and every polynomial $P \in \mathbb{P}_n$. □

3 Sobolev spaces with respect to measures

In this section we recall the definition of Sobolev spaces with respect to measures introduced in [36–38].

Definition 3.1 Given $1 \le p < \infty$ and a set A which is a union of intervals in $\mathbb{R}$, we denote by $B_p(A)$ the set of weights w such that $w^{-1} \in L^{1/(p-1)}_{loc}(A)$ (if $p = 1$, then $1/(p-1) = \infty$).

It is possible to construct a similar theory with $p = \infty$. We refer to [2,30–32] for the case $p = \infty$.

If $A = \mathbb{R}$, then $B_p(\mathbb{R})$ contains, as a very particular case, the classical $A_p(\mathbb{R})$ weights appearing in Harmonic Analysis. The classes $B_p(\Omega)$, with $\Omega \subseteq \mathbb{R}^n$, have been used in other definitions of weighted Sobolev spaces in $\mathbb{R}^n$ in [17].

We consider vector measures $\mu = (\mu_0,\ldots,\mu_k)$ in the definition of our Sobolev space in $\mathbb{R}$. We assume that each μ_j is σ-finite; hence, by Radon–Nikodym's Theorem, we have the decomposition $d\mu_j = d(\mu_j)_s + w_j ds$, where $(\mu_j)_s$ is singular with respect to Lebesgue measure and w_j is a non-negative Lebesgue measurable function.

In [17], Kufner and Opic define the following sets:

Definition 3.2 Let us consider $1 \le p < \infty$ and a vector measure $\mu = (\mu_0, \dots, \mu_k)$. For $0 \le j \le k$ we define the open set

$$\Omega_j := \left\{ x \in \mathbb{R} \ : \ \exists \text{ an open neighborhood } V \text{ of } x \text{ with } w_j \in B_p(V) \right\}.$$

Note that we always have $w_j \in B_p(\Omega_j)$ for any $0 \le j \le k$. In fact, Ω_j is the largest open set U with $w_j \in B_p(U)$. It is easy to check that if $f^{(j)} \in L^p(\Omega_j, w_j)$, $1 \le j \le k$, then $f^{(j)} \in L^1_{loc}(\Omega_j)$ and, therefore, $f^{(j-1)} \in AC_{loc}(\Omega_j)$, i.e., $f^{(j-1)}$ is a locally absolutely continuous function in Ω_j.

Since the precise definition of Sobolev space requires some technical concepts (see Definition 3.8), we would like to introduce here a heuristic definition of Sobolev space and an example which will help us to understand the technical process that we will follow in order to reach Definition 3.8.

Definition 3.3 (*Heuristic definition*) Let us consider $1 \le p < \infty$ and a p-admissible vector measure $\mu = (\mu_0, \dots, \mu_k)$ in $\mathbb{R}$. We define the *Sobolev space* $W^{k,p}(\mu) = W^{k,p}(\Delta, \mu)$, with $\Delta := \cup_{j=0}^k \operatorname{supp}(\mu_j)$, as the space of equivalence classes of

$$\begin{aligned} V^{k,p}(\mu) := V^{k,p}(\Delta,\mu) :=\Big\{ f : \Delta \to \mathbb{R} \ : \ & \|f\|_{W^{k,p}(\Delta,\mu)} := \left(\sum_{j=0}^{k} \|f^{(j)}\|^p_{L^p(\Delta,\mu_j)} \right)^{1/p} < \infty, \\ & f^{(j)} \in AC_{loc}(\Omega_{j+1} \cup \cdots \cup \Omega_k) \text{ and } f^{(j)} \text{ satisfies} \\ & \text{"pasting conditions" for } 0 \le j < k \Big\}, \end{aligned}$$

with respect to the seminorm $\| \cdot \|_{W^{k,p}(\Delta,\mu)}$.

These pasting conditions are natural: a function must be as regular as possible. In a first step, we check if the functions and their derivatives are absolutely continuous up to the boundary (this fact holds in the following example), and then we join the contiguous intervals:

Example $\mu_0 := \delta_0$, $\mu_1 := 0$, $d\mu_2 := \chi_{[-1,0]}(x)dx$ and $d\mu_3 := \chi_{[0,1]}(x)dx$, where χ_A denotes the characteristic function of the set A.

Since $\Omega_1 = \emptyset$, $\Omega_2 = (-1, 0)$ and $\Omega_3 = (0, 1)$, $W^{3,p}(\mu)$ is the space of equivalence classes of

$$\begin{aligned} V^{3,p}(\mu) =& \Big\{ f \ : \ \|f\|_{W^{3,p}(\mu)} < \infty\,, \ f, f' \text{ satisfy "pasting conditions"}, \\ & f, f', \in AC((-1,0)) \ \text{ and } \ f, f', f'' \in AC((0,1)) \Big\} \\ =& \Big\{ f \ : \ \|f\|_{W^{3,p}(\mu)} < \infty\,, \ f, f' \text{ satisfy "pasting conditions"}, \\ & f, f', \in AC([-1,0]) \ \text{ and } \ f, f', f'' \in AC([0,1]) \Big\} \\ =& \Big\{ f \ : \ \|f\|_{W^{3,p}(\mu)} < \infty\,, \ f, f' \in AC([-1,1]) \ \text{ and } \ f'' \in AC([0,1]) \Big\}. \end{aligned}$$

In the current case, since f and f' are absolutely continuous in $[-1, 0]$ and in $[0, 1]$, we require that both are absolutely continuous in $[-1, 1]$.

These heuristic concepts can be formalized as follows:

Definition 3.4 Let us consider $1 \le p < \infty$ and μ, ν measures in $[a, b]$. We define

$$\Lambda^{+}_{p,[a,b]}(\mu, \nu) := \sup_{a<x<b} \mu((a, x]) \left\| (d\nu/ds)^{-1} \right\|_{L^{1/(p-1)}([x,b])} ,$$
$$\Lambda^{-}_{p,[a,b]}(\mu, \nu) := \sup_{a<x<b} \mu([x, b)) \left\| (d\nu/ds)^{-1} \right\|_{L^{1/(p-1)}([a,x])} ,$$

where we use the convention $0 \cdot \infty = 0$.

Muckenhoupt inequality (See [28], [23, p. 44], [2, Theorem 3.1]). *Let us consider $1 \le p < \infty$ and μ_0, μ_1 measures in $[a, b]$. Then:*

(1) *There exists a real number c such that*

$$\left\| \int_x^b g(t)\, dt \right\|_{L^p((a,b],\mu_0)} \le c \left\| g \right\|_{L^p((a,b],\mu_1)}$$

for any measurable function g in $[a, b]$, if and only if $\Lambda^{+}_{p,[a,b]}(\mu_0, \mu_1) < \infty$.

(2) *There exists a positive constant c such that*

$$\left\| \int_a^x g(t)\, dt \right\|_{L^p([a,b),\mu_0)} \le c \left\| g \right\|_{L^p([a,b),\mu_1)}$$

for any measurable function g in $[a, b]$, if and only if $\Lambda^{-}_{p,[a,b]}(\mu_0, \mu_1) < \infty$.

Definition 3.5 Let us consider $1 \le p < \infty$. A vector measure $\overline{\mu} = (\overline{\mu}_0, \ldots, \overline{\mu}_k)$ is a right completion of a vector measure $\mu = (\mu_0, \ldots, \mu_k)$ in $\mathbb{R}$ with respect to a in a right neighborhood $[a, b]$, if $\overline{\mu}_k = \mu_k$ in $[a, b]$, $\overline{\mu}_j = \mu_j$ in the complement of $(a, b]$ and

$$\overline{\mu}_j = \mu_j + \tilde{\mu}_j , \quad \text{in } (a, b] \quad \text{for } 0 \le j < k ,$$

where $\tilde{\mu}_j$ is any measure satisfying $\tilde{\mu}_j((a, b]) < \infty$ and $\Lambda^{+}_{p,[a,b]}(\tilde{\mu}_j, \overline{\mu}_{j+1}) < \infty$.

Muckenhoupt inequality guarantees that if $f^{(j)} \in L^p(\mu_j)$ and $f^{(j+1)} \in L^p(\overline{\mu}_{j+1})$, then $f^{(j)} \in L^p(\overline{\mu}_j)$ (see some examples of completions in [2,36]).

Remark 3.1 We can define a left completion of μ with respect to a in a similar way.

Definition 3.6 For $1 \le p < \infty$ and a vector measure μ in $\mathbb{R}$, we say that a point a is right j-regular (respectively, left j-regular), if there exist a right completion $\overline{\mu}$ (respectively, left completion) of μ in $[a, b]$ and $j < i \le k$ such that $\overline{w}_i \in B_p([a, b])$ (respectively, $B_p([b, a])$). Also, we say that a point $a \in \gamma$ is j-regular, if it is right and left j-regular.

Remark 3.2 1. A point a is right j-regular (respectively, left j-regular), if at least one of the following properties holds:

(a) There exist a right (respectively, left) neighborhood $[a, b]$ (respectively, $[b, a]$) and $j < i \leq k$ such that $w_i \in B_p([a, b])$ (respectively, $B_p([b, a])$). Here we have chosen $\tilde{w}_j = 0$.

(b) There exist a right (respectively, left) neighborhood $[a, b]$ (respectively, $[b, a]$) and $j < i \leq k$, $\alpha > 0$, $\delta < (i - j)p - 1$, such that $w_i(x) \geq \alpha\, |x - a|^\delta$, for almost every $x \in [a, b]$ (respectively, $[b, a]$). See Lemma 3.4 in [36].

2. If a is right j-regular (respectively, left), then it is also right i-regular (respectively, left) for each $0 \leq i \leq j$.

When we use this definition we think of a point $\{t\}$ as the union of two half-points $\{t^+\}$ and $\{t^-\}$. With this convention, each one of the following sets

$$(a, b) \cup (b, c) \cup \{b^+\} = (a, b) \cup [b^+, c) \neq (a, c)\,,$$
$$(a, b) \cup (b, c) \cup \{b^-\} = (a, b^-] \cup (b, c) \neq (a, c)\,,$$

has two connected components, and the set

$$(a, b) \cup (b, c) \cup \{b^-\} \cup \{b^+\} = (a, b) \cup (b, c) \cup \{b\} = (a, c)$$

is connected.

We use this convention in order to study the sets of continuity of functions: we want that if $f \in C(A)$ and $f \in C(B)$, where A and B are union of intervals, then $f \in C(A \cup B)$. With the usual definition of continuity, if $f \in C([a, b)) \cap C([b, c])$ then we do not have $f \in C([a, c])$. Of course, we have $f \in C([a, c])$ if and only if $f \in C([a, b^-]) \cap C([b^+, c])$, where by definition, $C([b^+, c]) = C([b, c])$ and $C([a, b^-]) = C([a, b])$. This idea can be formalized with a suitable topological space.

Let us introduce some more notation. We denote by $\Omega^{(j)}$ the set of j-regular points or half-points, i.e., $x \in \Omega^{(j)}$ if and only if x is j-regular, we say that $x^+ \in \Omega^{(j)}$ if and only if x is right j-regular, and we say that $x^- \in \Omega^{(j)}$ if and only if x is left j-regular. Obviously, $\Omega^{(k)} = \emptyset$ and $\Omega_{j+1} \cup \cdots \cup \Omega_k \subseteq \Omega^{(j)}$. Note that $\Omega^{(j)}$ depends on p.

Intuitively, $\Omega^{(j)}$ is the set of "good" points at the level j for the vector weight $(w_0, \ldots, w_k)$: every function f in the Sobolev space must verify that $f^{(j)}$ is continuous in $\Omega^{(j)}$.

Let us present now the class of measures that we use in the definition of Sobolev space.

Definition 3.7 We say that the vector measure $\mu = (\mu_0, \ldots, \mu_k)$ in $\mathbb{R}$ is *p-admissible* if μ_j is σ-finite and $\mu_j^*(\mathbb{R}\backslash\Omega^{(j)}) = 0$, for $1 \leq j < k$, and $\mu_k^* \equiv 0$, where $d\mu_j^* := d\mu_j - w_j \chi_{\Omega_j} dx$ and χ_A denotes the characteristic function of the set A (then $d\mu_k = w_k \chi_{\Omega_k} dx$).

Remark 3.3 1. The hypothesis of p-admissibility is natural. It would not be reasonable to consider Dirac's deltas in μ_j in the points where $f^{(j)}$ is not continuous.

2. Note that there is not any restriction on μ_0.

3. Every absolutely continuous measure $w = (w_0, \dots, w_k)$ with $w_j = 0$ a.e. in $\mathbb{R}\setminus\Omega_j$ for every $1 \le j \le k$, is p-admissible (since then $\mu_j^* = 0$). It is possible to find a weight w which does not satisfy this condition, but it is a hard task.
4. $(\mu_j)_s \le \mu_j^*$, and the equality usually holds.

Definition 3.8 Let us consider $1 \le p < \infty$ and a p-admissible vector measure $\mu = (\mu_0, \dots, \mu_k)$ in $\mathbb{R}$. We define the *Sobolev space* $W^{k,p}(\mu) = W^{k,p}(\Delta, \mu)$, with $\Delta := \cup_{j=0}^k \mathrm{supp}(\mu_j)$, as the space of equivalence classes of

$$V^{k,p}(\Delta, \mu) := \Bigg\{ f : \Delta \to \mathbb{R} \ : \ f^{(j)} \in AC_{loc}(\Omega^{(j)}) \text{ for } 0 \le j < k \text{ and } \|f\|_{W^{k,p}(\Delta,\mu)} := \left(\sum_{j=0}^k \left\| f^{(j)} \right\|_{L^p(\Delta,\mu_j)}^p \right)^{1/p} < \infty \Bigg\},$$

with respect to the seminorm $\|\cdot\|_{W^{k,p}(\Delta,\mu)}$.

4 Basic results on Sobolev spaces with respect to measures

This definition of Sobolev space is very technical, but it has interesting properties: we know explicitly how are the functions in $W^{k,p}(\Delta, \mu)$ (this is not the case if we define the Sobolev space as the closure of some space of smooth functions, as in [19–21]); if Δ is a compact set and μ is a finite measure, then in many cases, $W^{k,p}(\Delta, \mu)$ is equal to the closure of the space of polynomials (see [2, Theorem 6.1]). Furthermore, we have powerful tools in $W^{k,p}(\Delta, \mu)$ (see [2,36,37,39]).

In [39, Theorem 4.2] appears the following main result in the theory (in fact, this result in [39] holds for measures defined in any curve in the complex plane instead of $\mathbb{R}$).

Theorem 4.1 *Let us consider* $1 \le p < \infty$ *and a* p*-admissible vector measure* $\mu = (\mu_0, \dots, \mu_k)$*. Then the Sobolev space* $W^{k,p}(\Delta, \mu)$ *is a Banach space.*

We want to remark that the proof of Theorem 4.1 is very long and technical: the paper [36] is mainly devoted to prove a weak version of Theorem 4.1, and using this version, [39] provides a very technical proof of the general case. Note that it took 6 years to prove this natural property of Sobolev spaces with respect to measures.

For each $1 \le p < \infty$ and p-admissible vector measure μ in $\mathbb{R}$, consider the Banach space $\prod_{j=0}^k L^p(\Delta, \mu_j)$ with the norm

$$\|f\|_{\prod_{j=0}^k L^p(\Delta,\mu_j)} = \left(\sum_{j=0}^k \|f_j\|_{L^p(\mu_j)}^p \right)^{1/p}$$

for every $f = (f_0, f_1, \dots, f_k) \in \prod_{j=0}^k L^p(\Delta, \mu_j)$.

Definition 4.1 For each $1 \leq p < \infty$ we denote by q the conjugate or dual exponent of p, i.e., $1/p + 1/q = 1$. Consider a p-admissible vector measure μ. If $f \in \prod_{j=0}^{k} L^p(\Delta, \mu_j)$ and $g \in \prod_{j=0}^{k} L^q(\Delta, \mu_j)$, we define the product (f, g) as

$$(f, g) = \sum_{j=0}^{k} \int_{\Delta} f_j \, g_j \, d\mu_j \, .$$

Let us consider the projection $P : W^{k,p}(\Delta, \mu) \longrightarrow \prod_{j=0}^{k} L^p(\Delta, \mu_j)$, given by $Pf = (f, f', \ldots, f^{(k)})$.

Theorem 4.2 *Let $1 \leq p < \infty$ and μ a p-admissible vector measure. Then $W^{k,p}(\Delta, \mu)$ is separable. Furthermore, if $1 < p < \infty$, then it is reflexive and uniformly convex.*

Proof The map P is an isometric embedding of $W^{k,p}(\Delta, \mu)$ onto $W := P(W^{k,p}(\Delta, \mu))$. Then W is a closed subspace since $\prod_{j=0}^{k} L^p(\Delta, \mu_j)$ and $W^{k,p}(\Delta, \mu)$ are Banach spaces by Theorem 4.1.

If $1 \leq p < \infty$, then each $L^p(\Delta, \mu_j)$ is separable; furthermore, if $1 < p < \infty$, then it is reflexive and uniformly convex. Then $\prod_{j=0}^{k} L^p(\Delta, \mu_j)$ is separable (and reflexive and uniformly convex if $1 < p < \infty$) by [1, p. 8].

Since W is closed and $\prod_{j=0}^{k} L^p(\Delta, \mu_j)$ is separable, W is separable; furthermore, if $1 < p < \infty$, then W is reflexive and uniformly convex since $\prod_{j=0}^{k} L^p(\Delta, \mu_j)$ is reflexive and uniformly convex (see [1, p. 7], [14]). Since W and $W^{k,p}(\Delta, \mu)$ are isometric, $W^{k,p}(\Delta, \mu)$ also has these properties.

Theorem 4.3 *Let $1 \leq p < \infty$, q the dual exponent of p, and μ a p-admissible vector measure. Consider the canonical map $J : \prod_{j=0}^{k} L^q(\Delta, \mu_j) \longrightarrow (W^{k,p}(\Delta, \mu))'$ defined by $J(v) = (\cdot, v)$, i.e., $\big(J(v)\big)(f) = (Pf, v)$. Then giving any $T \in (W^{k,p}(\Delta, \mu))'$ there exists $v \in \prod_{j=0}^{k} L^q(\Delta, \mu_j)$ with*

$$T = J(v) \quad \textit{and} \quad \big\|T\big\|_{(W^{k,p}(\Delta,\mu))'} = \|v\|_{\prod_{j=0}^{k} L^q(\Delta,\mu_j)} . \tag{4.1}$$

Furthermore, if $1 < p < \infty$, then there exists a unique $v \in \prod_{j=0}^{k} L^q(\Delta, \mu_j)$ verifying (4.1).

Proof First of all we will prove that J is, in fact, a map $J : \prod_{j=0}^{k} L^q(\Delta, \mu_j) \longrightarrow (W^{k,p}(\Delta, \mu))'$. Given $v \in \prod_{j=0}^{k} L^q(\Delta, \mu_j)$, consider $J(v)$. Then continuous and discrete Hölder's inequalities give

$$
\begin{aligned}
\left|\big(J(v)\big)(f)\right| = \left|(Pf, v)\right| &\le \sum_{j=0}^{k} \int_{\Delta} \left| f^{(j)}\, v_j \right| d\mu_j \le \sum_{j=0}^{k} \left\| f^{(j)} \right\|_{L^p(\mu_j)} \left\| v_j \right\|_{L^q(\mu_j)} \\
&\le \left(\sum_{j=0}^{k} \left\| f^{(j)} \right\|_{L^p(\mu_j)}^{p} \right)^{1/p} \left(\sum_{j=0}^{k} \left\| v_j \right\|_{L^q(\mu_j)}^{q} \right)^{1/q} \\
&= \| f \|_{W^{k,p}(\Delta,\mu)} \| v \|_{\prod_{j=0}^{k} L^q(\Delta,\mu_j)}.
\end{aligned}
$$

Hence,

$$
\left\| J(v) \right\|_{(W^{k,p}(\Delta,\mu))'} \le \| v \|_{\prod_{j=0}^{k} L^q(\Delta,\mu_j)}. \tag{4.2}
$$

Thus we have proved that $J : \prod_{j=0}^{k} L^q(\Delta, \mu_j) \longrightarrow (W^{k,p}(\Delta, \mu))'$. Let us prove now that J is onto.

Consider $T \in (W^{k,p}(\Delta, \mu))'$. The map P is an isometric isomorphism of $W^{k,p}(\Delta, \mu)$ onto $W := P(W^{k,p}(\Delta, \mu))$. Then $T \circ P^{-1} \in W'$ and

$$
\left\| T \circ P^{-1} \right\|_{W'} = \left\| T \right\|_{\left(W^{k,p}(\Delta,\mu)\right)'}.
$$

Since W is a subspace of $\prod_{j=0}^{k} L^p(\Delta, \mu_j)$, by Hahn–Banach Theorem there exists

$$
T_0 \in \left(\prod_{j=0}^{k} L^p(\Delta, \mu_j) \right)' = \prod_{j=0}^{k} \left(L^p(\Delta, \mu_j) \right)' = \prod_{j=0}^{k} L^q\left(\Delta, \mu_j\right),
$$

with

$$
T_0|_{W'} = T \circ P^{-1}, \quad \left\| T_0 \right\|_{\left(\prod_{j=0}^{k} L^p(\Delta,\mu_j)\right)'} = \left\| T \circ P^{-1} \right\|_{W'}.
$$

Therefore, there exists $v \in \prod_{j=0}^{k} L^q(\Delta, \mu_j)$ with $v \simeq T_0$, i.e.,

$$
T_0(f) = (f, v), \quad \forall f \in W,
$$

and

$$
\| v \|_{\prod_{j=0}^{k} L^q(\Delta,\mu_j)} = \left\| T_0 \right\|_{\left(\prod_{j=0}^{k} L^p(\Delta,\mu_j)\right)'} = \left\| T \circ P^{-1} \right\|_{W'} = \left\| T \right\|_{\left(W^{k,p}(\Delta,\mu)\right)'}. \tag{4.3}
$$

Hence,

$$
T(f) = T_0(Pf) = (Pf, v) = \big(J(v)\big)(f), \quad \forall f \in W^{k,p}(\Delta, \mu),
$$

$T = J(v)$ and J is onto.

If $1 < p < \infty$, let us consider the set $U := \{u \in \prod_{j=0}^{k} L^q(\Delta, \mu_j) \,:\, T = J(u)\}$. Note that (4.2) and (4.3) give

$$\|T\|_{(W^{k,p}(\Delta,\mu))'} = \inf\left\{\|u\|_{\prod_{j=0}^{k} L^q(\Delta,\mu_j)} \,:\, u \in U\right\}$$
$$= \min\left\{\|u\|_{\prod_{j=0}^{k} L^q(\Delta,\mu_j)} \,:\, u \in U\right\}.$$

It is easy to check that U is a closed convex set in $\prod_{j=0}^{k} L^q(\Delta, \mu_j)$. Since $\prod_{j=0}^{k} L^q(\Delta, \mu_j)$ is uniformly convex, this minimum is attained at a unique $u_0 \in U$ (see, e.g., [6, p. 22]), and (4.3) gives $u_0 = v$.

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